Exercise IX, Theory of Computation 2025

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

These problems are taken from various sources at EPFL and on the Internet, too numerous to cite individually.

1 A k-CNF formula is a CNF formula where each clause has at most k literals. The language $k\mathsf{SAT}$ consists of all k-CNF formulas which are satisfiable. What is wrong with the following proof of **NP**-completeness of $2\mathsf{SAT}$?

Since $3\mathsf{SAT}$ *is in* **NP**, *so is* $2\mathsf{SAT}$. *For any 2-CNF* $\varphi = (a_1 \vee b_1) \wedge \cdots \wedge (a_n \vee b_n)$, *define the 3-CNF* $f(\varphi) = (a_1 \vee b_1 \vee b_1) \wedge \cdots \wedge (a_n \vee b_n \vee b_n)$, *where the* a_i *and* b_i *are literals. Note that*

$$\varphi \text{ is satisfiable} \iff f(\varphi) \text{ is satisfiable.}$$

Hence $3\mathsf{SAT} \leq_P 2\mathsf{SAT}$, *and it follows that* $2\mathsf{SAT}$ *is* **NP**-*complete.*

2 Prove that the following problem is **NP**-complete: Given a set S, a collection $\mathcal{C}$ of subsets of S and an integer k, is there a subset $T \subseteq S$ of size at most k such that $T \cap C \neq \emptyset$ for all $C \in \mathcal{C}$?

3 Using the **NP**-completeness $\mathsf{SubsetSum}$, prove that the following problem is **NP**-complete: Given integers $V, v_1, \ldots, v_n$ and $W, w_1, \ldots, w_n$, is there a subset S of $\{1, 2, \ldots, n\}$ such that

$$\sum_{i \in S} w_i \leq W \text{ and } \sum_{i \in S} v_i \geq V?$$

4* Prove that the following problem is **NP**-hard: Given vectors $v_1, \ldots, v_m, V \in \mathbb{Z}^n$ and $K \in \mathbb{Z}$, do there exist $a_1, \ldots, a_m \in \mathbb{Z}$ such that

$$\left\| \sum_{i=1}^{m} a_i v_i - V \right\|^2 \leq K?$$

Note that for a vector $v \in \mathbb{R}^n$, we have $\|v\|^2 = \sum_{i=1}^n v_i^2$.

Hint: *First consider the variant of the problem where* $a_1, a_2, \ldots, a_m \in \{0, 1\}$.